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Symplectic Generating Functions and Moyal Products

PAVOL ŠEVERA[1,2]
[1] *Section de Mathématiques, Université de Genève, Rue du Lièvre 2-4, 1211 Genève, Switzerland. e-mail:pavol.severa@gmail.com*
[2] *Department of Theoretical Physics, Commenius University, Mlynská dolina F2, 84248 Bratislava, Slovakia*

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Abstract. We introduce an affine-invariant version of generating functions of symplectic transformations of affine symplectic spaces, together with a generalization for other symmetric symplectic spaces. The composition of these functions has a nice connection with the Moyal product.

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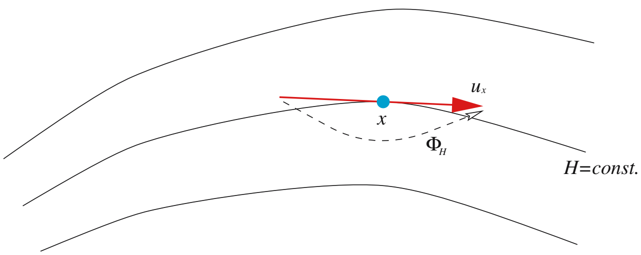
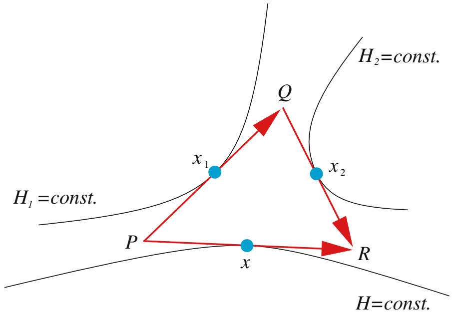
Let us start with the simplest, but probably the most useful result of this note. Suppose A is an affine symplectic space. There is an affine-invariant way in which functions on A generate symplectic transformations of A. Namely, let H be a function on A. At any point $x \in A$ we take the vector u_x defined by $d_x H = \omega(\cdot, u_x)$ (where ω is the symplectic form) and substitute it in A so that x lies in its middle. Then the map Φ_H sending the tails of u_x's to their heads is a symplectic transformation:



Notice that for infinitesimal Hamiltonians, this is the usual infinitesimal transformation generated by the Hamiltonian H. The map $H \mapsto \Phi_H$ is a kind of Cayley transform: choosing an origin in A to make it to a vector space, and restricting ourselves to H's which are quadratic forms, we get the usual Cayley transform $\mathfrak{sp} \to Sp$.

Symplectic transformation can, of course, be composed. The corresponding composition of generating functions is

$$H(x) = H_1(x_1) + H_2(x_2) + \text{symplectic area of } \triangle PQR:$$



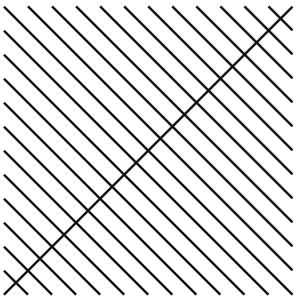
This composition law is closely related with the integral kernel of the Moyal product,

$$K(x_1, x_2, x) = \exp(\sqrt{-1} \times \text{symplectic area of } \triangle PQR/\hbar):$$

the function $\exp(\sqrt{-1}H/\hbar)$ is the leading oscillatory part of the Moyal product of $\exp(\sqrt{-1}H_1/\hbar)$ with $\exp(\sqrt{-1}H_2/\hbar)$.

The composition law appeared previously in [2], which was inspired earlier in [1], which should refer to even earlier [5]. More on the connection with quantization can be found in [4].

Let us prove our claims. The graph of any symplectic transformation of A is a Lagrangian submanifold of $\bar{A} \times A$. For each point $x \in A$ the symmetry with respect to x is a symplectic transformation. Their graphs, together with the graph of the identity, can be used to identify $\bar{A} \times A$ with T^*A:

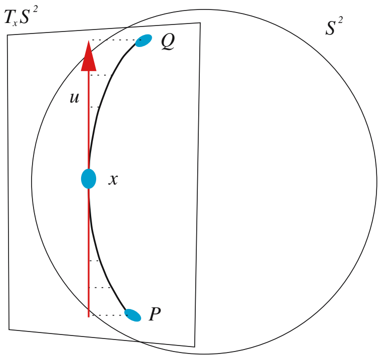


Explicitly (as one immediately sees from the picture), a pair $(P, Q) \in \bar{A} \times A$ corresponds to $((P+Q)/2, \omega(\cdot, Q-P)) \in T^*A$. Here the vector-and-its-midpoint picture emerges.

The correspondence between generating functions and symplectic transformations is clear now: dH is a Lagrangian submanifold of T^*A, and therefore of $\bar{A}\times A$; if it is the graph of some transformation of A, the transformation is symplectic.

Let us explain where the composition law for generating functions come from. The space $\Gamma=\bar{A}\times A$ is a symplectic groupoid (the pair groupoid of A). The graph of its multiplication is a Lagrangian submanifold in $\Gamma\times\Gamma\times\bar{\Gamma}$; using the identification of T^*A with $\Gamma=\bar{A}\times A$, it should be given by a closed 1-form on $A\times A\times A$. This 1-form is the differential of the function $(x_1,x_2,x)\mapsto$ symplectic area of $\triangle PQR$ (this was observed in much greater generality by Alan Weinstein in [6]; we shall need his more general statement later). The composition law and its connection with Moyal product are now clear.

For the fun of it, let us make a similar construction, replacing A by the sphere S^2 with its area symplectic form. Again symmetry with respect to a point of S^2 (i.e., rotation of the sphere by π around the axis passing though that point) is a symplectic transformation of the sphere, therefore we locally have a similar identification between $\overline{S^2}\times S^2$ and T^*S^2; more precisely, there is an isomorphism between the subset of covectors in T^*S^2 of length less than 2 and $\overline{S^2}\times S^2$ with erased pairs of antipodal points. Explicitly, to a non-antipodal pair (P,Q) we associate a $u\in TS^2$ (and thus, via ω, a point in T^*S^2) as on the picture:



Here x is the midpoint of the short geodesic arc PQ and $u\in T_xS^2$ appears by the orthogonal projection of the arc. This picture can be derived from the famous theorem of Archimedes, claiming that a certain map between the cylinder and sphere is area-preserving.

As a result, we have a similar picture of generating functions: for a function H on S^2 and any point $x\in S^2$ we take the vector u_x defined by $d_xH=\omega(\cdot,u_x)$, place it into the tangent plane T_xS^2 so that x is in its middle and project it into the sphere; Φ_H maps P to Q. Composition rule looks as before, only the triangles are spherical now (as follows again from [6]).

Generally, this picture works with no changes for an arbitrary symmetric symplectic space M: we embed M into an affine space A (just as S^2 is embedded into $\mathbb{R}^3$) and extend the symmetry of M with respect to any point $x \in M$ to an involution σ_x of A. Then we project M to $T_x M$ in the direction of A^{σ_x} (just as we used the orthogonal projection of the sphere to its tangent planes). The generation of symplectic transformations of M by functions on M, and the composition law for generating functions, are then the same as for the sphere (this was also noticed in [3], in a somewhat different way).

Here are the details: Since M is a homogeneous space of some Lie group G, we can find a moment map $M \to \mathfrak{g}^*$ (the map is possibly non-equivariant and defined only on a covering of M; in fact, being a homogeneous symplectic space, M is, up to covering, a coadjoint orbit of a central extension of G). This is the immersion $M \to A$ we need. Now, since M is a symmetric space, it is (up to covering) G/G^σ, where σ is an involutory automorphism of G. Let $\mathfrak{g} = \mathfrak{g}^\sigma \oplus \mathfrak{p}$ be the decomposition of $\mathfrak{g}$ to ± 1 eigenspaces of $d\sigma$; to make G/G^σ to a symmetric symplectic space, we have to choose a $\mathfrak{g}^\sigma$-invariant symplectic form on $\mathfrak{p}$. If $x \in M$ is fixed by G^σ (for other x values we have to conjugate σ with the corresponding inner automorphism of G), $T_x M$ is $\mathfrak{p}^*$ translated to x; we project M to $T_x M$ in the direction of $(\mathfrak{g}^*)^\sigma$.

To finish, we have to prove that these projections $M \to T_x M$ actually give us a local symplectomorhism between $\bar{M} \times M$ and $T^* M$. Let us denote the projection by $\pi : M \to T_x M$. Let us choose $t \in \mathfrak{p}$ and $u \in T_x M = \mathfrak{p}^*$; t gives us a vector field on M, its value at $y \in M$ will be denoted by t_y. We have to prove that for any $\lambda \in \mathbb{R}$ we have

$$\omega_x(u, t_x) = \omega_{\pi^{-1}(\lambda u)}(\pi_*^{-1} u, t_{\pi^{-1}(\lambda u)});$$

this will ensure that the fundamental 1-form on $T^* M$ is mapped to the primitive of the symplectic form on $\bar{M} \times M$ given by its Lagrangian fibration. To prove the formula, we just notice that its RHS is equal to $\langle \pi_*^{-1} u, t \rangle$, which is in turn equal to $\langle u, t \rangle$ (since we project in a direction that is in the kernel of t), which is the LHS.

Acknowledgements

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